March 14, 2017

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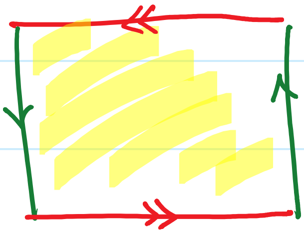
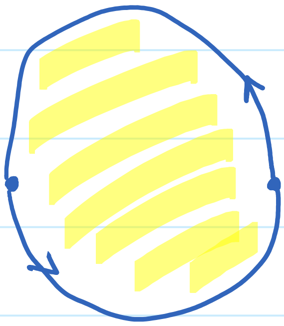
More on Projective Plane $\mathbb{RP}^2$

It can be constructed in these ways

* 

$([0,1]\times[0,1])/\sim$ by identifying $(0,s)$ with $(1,1-s)$ and $(t,0)$ with $(1-t,1)$

* 

$\mathbb{D}^2/\sim$ by identifying $e^{i\theta}$ with $-e^{i\theta}$ on S^1

Space of st. lines thru $(0,0,0)\in\mathbb{R}^3$

Define $\sim$ on $\mathbb{R}^3\setminus\{\vec{0}\}$ by

$$\vec{x}\sim\vec{y} \text{ if } \underbrace{\exists\, 0\neq\lambda\in\mathbb{R} \quad \vec{x}=\lambda\vec{y}}_{\vec{x},\vec{0},\vec{y} \text{ lie on the same st. line}}$$

Then

$$[\vec{x}]\in(\mathbb{R}^3\setminus\{\vec{0}\})/\sim$$
$$\|$$
$$\{\lambda\vec{x} : 0\neq\lambda\in\mathbb{R}\} \text{ is a st. line thru } \vec{0}$$

* On $S^2=\{\vec{u}\in\mathbb{R}^3 : \|\vec{u}\|=1\}$,

$\vec{u}$ and $-\vec{u}$ are called antipodal points.

Let $S^2/\approx$ be identifying antipodal points

Intuitively, one suspects $S^2/\approx \longleftrightarrow (\mathbb{R}^3\setminus\{\vec{0}\})/\sim$

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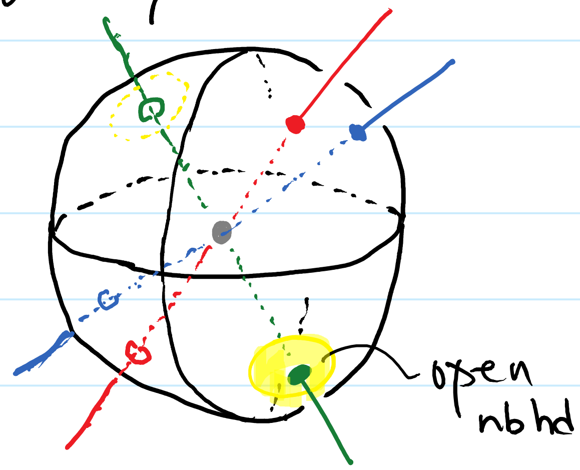
$$S^2/\text{antipodal points} \xleftrightarrow{\text{homeo}} (\mathbb{R}^3\setminus\{\vec{0}\})/\vec{x}\sim\lambda\vec{y}$$

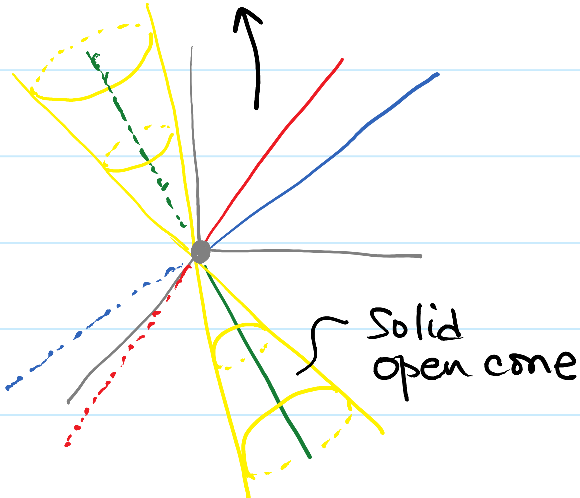
$$[\vec{u}]=\{\vec{u},-\vec{u}\} \longrightarrow [\vec{u}]=\{\lambda\vec{u} : 0\neq\lambda\in\mathbb{R}\}$$

$$\left[\frac{\vec{x}}{\|\vec{x}\|}\right] \longleftarrow\!\!\mid [\vec{x}]$$

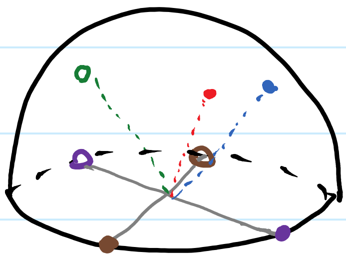
The following picture illustrates the continuity.

quotient $\nearrow$ $S^2/\text{antipodal} \longleftrightarrow (\mathbb{R}^3\setminus\{0\})/(\text{same line})$





$[u]=\{\vec{u},-\vec{u}\}$ The above is further related by the mapping below



$\subset \mathbb{R}^3$ (x_1,x_2,x_3)

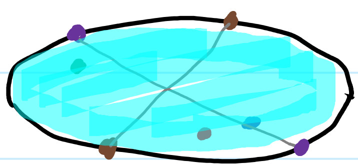
$\downarrow \pi$

$(x_1,x_2,0)$

$[\pi(\vec{u})]$

$\in$

$D^2/\sim \;=$



$\subset \mathbb{R}^2$

Real Projective Spaces $\mathbb{R}P^n$

Similarly, for $\vec{x}, \vec{y} \in \mathbb{R}^{n+1}$, define

$$\vec{x} \sim \vec{y} \text{ if } \exists\, 0 \neq \lambda \in \mathbb{R},\ \vec{x} = \lambda \vec{y}$$

$\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{\vec{0}\})/\sim$ is the space of st. lines thru $\vec{0}$ in $\mathbb{R}^{n+1}$

$$\mathbb{R}P^n = S^n/\text{antipodal points}$$

Exercise. $\mathbb{R}P^1$ is homeomorphic to S^1

Complex Projective Spaces $\mathbb{C}P^n$

For $z = (z_1, z_2, \cdots, z_{n+1}) \in \mathbb{C}^{n+1}$ and $w \in \mathbb{C}^{n+1}$,

$$z \sim w \text{ if } \exists\, 0 \neq \lambda \in \mathbb{C},\ z = \lambda w$$

$$\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$$

Note that this space has dimension $2n$.

One may also work on

$$S^{2n+1} = \{u \in \mathbb{C}^{n+1} : \|u\| = 1\}$$

In that case, $u, w \in S^{n+1}$ and $u \sim w$ if $\exists\, \theta \in \mathbb{R},\ u = e^{i\theta} w$

Exercise. $\mathbb{C}P^1 = S^2$.

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Matrix Quotient

Denote $O(n) = O_n(\mathbb{R}) = \{n\times n \text{ orthogonal real matrices}\}$

$= \{Q : Q^TQ = QQ^T = I\}$

Note that $O(n-1) \hookrightarrow O(n)$ by

$$Q \longmapsto \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & Q \\ 0 & \end{bmatrix}$$

Denote $O_3/O_2 = \{\underbrace{A\cdot O_2}_{\text{left cosets}} : A \in O_3\}$

This is a partition of O_3. In other words.

$A \sim B$ if $A\cdot O_2 = B\cdot O_2$

Equivalently, $A^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & O_2 & \end{bmatrix}$

In this case $A^{-1}B(e_1) = e_1$, i.e., $A(e_1) = B(e_1)$

We have the well-defined mapping

$$O_3/O_2 \longrightarrow S^2 : A\cdot O_2 \longmapsto A(e_1)$$

O_3 is given the standard topology as a subspace of $\mathbb{R}^9$. Then

O_3/O_2 is homeomorphic to S^2

Similarly, $S^n = \dfrac{O_{n+1}}{O_n}$

Projective Spaces.

Following the above, denote $\pm O_2 \subset O_3$ containing 3x3 matrices of the form

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{}} \\ 0 & & \end{bmatrix} \longleftarrow \text{2x2 orthogonal matrix}$$

Also, $O_3/(\pm O_2) = \{ A \cdot (\pm O_2) : A \in O_3 \}$ is given the quotient topology of $\mathbb{R}^9$

Now, if $A(\pm O_2) = B(\pm O_2)$ then $A(e_1) = \pm B(e_1)$ and $O_3/(\pm O_2) \longrightarrow \mathbb{R}P^2$ is a homeomorphism.

Similarly, $O_{n+1}/(\pm O_n) = \mathbb{R}P^n$ and

$U_{n+1}/(\pm U_n) = \mathbb{C}P^n$ where

$U_n(\mathbb{C}) = \{ n \times n \text{ complex unitary matrices} \}$

Grassmannian

Note that $\mathbb{R}P^n$ is the space of 1-dim vector spaces in $\mathbb{R}^{n+1}$. Using similarly method,

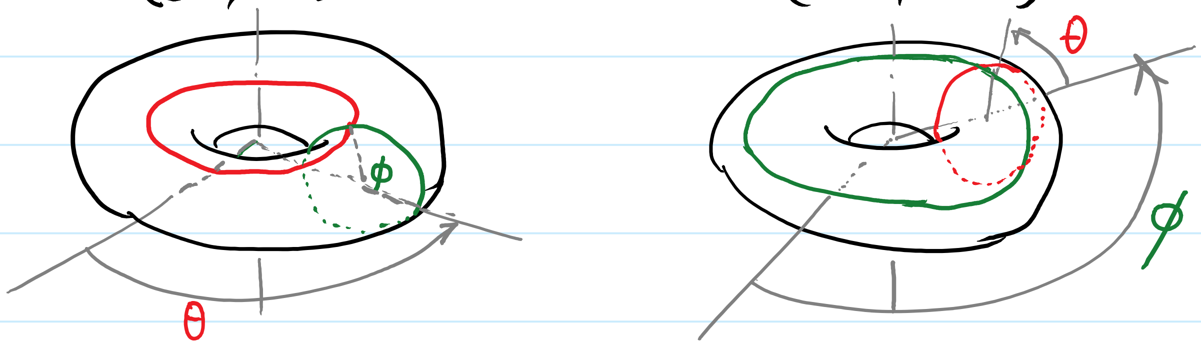
$G(n,k) = \dfrac{O_n}{O_{n-k} \times O_k}$ is the topological space of all k-dim vector subspaces in $\mathbb{R}^n$

Example. $S^3 = \{x=(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 : \|x\|=1\}$

Denote $S^1\times D^2$ a solid torus

$S^1 = \{e^{i\theta} : \theta\in\mathbb{R}\}$, $\quad D^2 = \{re^{i\phi} : \phi\in\mathbb{R},\ r\in[0,1]\}$

$$f: S^1\times S^1 \subset S^1\times D^2 \longrightarrow S^1\times D^2$$
$$(e^{i\theta}, e^{i\phi}) \longmapsto (e^{i\phi}, e^{i\theta})$$



According to f, a family of red longitudinal circles will be mapped to meridinal circles and green circles vice versa.

Illustration: $(S^1\times D^2)\cup_f(S^1\times D^2) = S^3 = \mathbb{R}^3\cup\{\infty\}$